S^3CS 2014 - Sum of Squares - Homework 1.

Exercise 1. The notion of pseudo-distributions may be a bit counter-intuitive. This exercise attempts to motivate it by showing that it is in some sense inherently needed whenever trying to use a convex relaxation to solve a combinatorial search problem. Any combinatorial search problem can be thought of as follows. We have a universe Ω (typically isomorphic to $\{\pm 1\}^n$) and a number of conditions $\mathcal{E}$ that implicitly define a set $S \subseteq \Omega$. The search task is find an element of S if it is not empty. An algorithm to solve the problem consists of a map $f : \Omega \to \tilde{\Omega}$ where $\tilde{\Omega}$ is some convex domain (typically $\mathbb{R}^m$ for some m) and a way to translate the condition $\mathcal{E}$ into conditions that define a *convex* set $\tilde{S} \subseteq \tilde{\Omega}$. The algorithm is a *valid relaxation* if for every $s \in S$, $f(s)$ is in $\tilde{S}$.

1. Define a mapping F that maps any *distribution* $\mathcal{D}$ on elements of Ω into a value $F(\mathcal{D}) \in \tilde{\Omega}$ such that:
   (a) If $\mathcal{D}$ is supported only on elements of S then $F(\mathcal{D})$ is in $\tilde{S}$.
   (b) Conclude that if there is an algorithm A that given an element in $\tilde{S}$ finds an element in S, then A must also be able to return an element of S given restricted access to a distribution $\mathcal{D}$ over S of the following form: A only gets the value $F(\mathcal{D})$.
2. Explain what are Ω, $\tilde{S}$, f, F in the case that S is defined by a set of polynomial equations that include the constraints $\{x_i^2 = 1\}$ and the relaxation is the degree d SOS program.

Exercise 2. Prove that if $\{x\}$ is a pseudo-distribution of degree d and P is a polynomial of degree d' then $\{P(x)\}$ is a pseudo-distribution of degree $\lfloor d/d' \rfloor$.

Exercise 3. Prove the pseudo-distribution Cauchy-Schwarz condition: If $\{x\}$ is a pseudo-distributionof degree at most $2d$, and P, Q are polynomials of degree d

$$\tilde{\mathbb{E}} PQ \leq \sqrt{\tilde{\mathbb{E}} P^2}\sqrt{\tilde{\mathbb{E}} Q^2}$$

Exercise 4. Let $P : \mathbb{R}^n \to \mathbb{R}$ of degree $\leq n$ such that $P(x) \geq 0$ for every $x \in \{\pm 1\}^n$.

1. Prove that there is an SOS proof of degree at most $2n$ that one cannot satisfy the equations $\{P(x) = -1\}$, $\{x_i^2 = 1\}_{i=1}^n$.
2. (Squared triangle inequality for $\{\pm 1\}$) Let $\{x\}$ be a pseudo-distribution of degree 6 satisfying the constraints $\{x_i^2 = 1\}$ for all i. Prove that for every i, j, k

$$\tilde{\mathbb{E}}(x_i - x_k)^2 \leq \tilde{\mathbb{E}}(x_i - x_j)^2 + \tilde{\mathbb{E}}(x_j - x_k)^2$$

Exercise 5.

1. Let P be a homogenous polynomial of degree at most 4. Prove that P is SOS if and only if $\tilde{\mathbb{E}} P(x) \geq 0$ for every degree 4 pseudo-distribution $\{x\}$.

2. Show an example of a pseudo-distribution $\{x\}$ and a polynomial P such that $\tilde{\mathbb{E}}P(x) = 0$ but $\{x\}$ does not satisfy the constraint $\{P(x) = 0\}$.

3. Show an example of a degree 4 pseudo-distribution $\{x\}$ such that there does not exist an actual distribution $\{y\}$ such that $\tilde{\mathbb{E}}P(x) = \mathbb{E}P(y)$ for every polynomial P of degree at most 4.

4. Prove that if $\{x\}$ is a degree $2d$ pseudo-distribution and P is a polynomial of degree at most d such that $\tilde{\mathbb{E}}P^2(x) = 0$ then $\{x\}$ is also a degree d pseudo-distribution that satisfies the constraint $\{P(x) = 0\}$.