# Expander Graphs and their Applications*
# Lecture 14: Some Eigenvalue Theorems

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**Summary:** The last lecture in the course surveys several bounds on eigenvalues of symmetric matrices, and in particular those of graphs. Throughout this summary, the eigenvalues of a graph refer to the eigenvalues of its adjacency matrix, and are denoted by $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$.

The class ended in a picnic, where juicy watermelon slices were served on Harry Potter plates, along side Harry Potter napkins, to students wearing silly Harry Potter paper hats.

**Theorem 1.** *For every $d$ and $\epsilon$ there is a $c = c(\epsilon, d)$, such that if $G$ is a $d$-regular graph on $n$ vertices, then the number of its eigenvalues with absolute value greater than $2\sqrt{d-1} - \epsilon$ is at least $cn$.*

*Proof.* See "Elementary Number Theory, Group Theory and Ramanujan Graphs", by G. Davidoff, P. Sarnak and A. Vallete. The proof makes use of the notorious Čebyshev Polynomials. □

**Theorem 2.** *(Füredi & Komloš '81) : Let $P_1$ be a random distribution on $\mathbb{R}$ with expectation $\mu$ and variance $\sigma^2$, and $P_2$ a random distribution on $\mathbb{R}$ with expectation $\nu$ and variance $\sigma^2$ as well. Assume further that both distributions are bounded. Let $A$ be a real, $n \times n$ symmetric matrix, with off-diagonal entries chosen i.i.d. according to $P_1$, and diagonal entries chosen i.i.d. according to $P_2$. Then with probability tending to $1$ as $n$ tends to infinity the following holds:*

1. $\max_{i \geq 2} |\lambda_i| < 2\sigma n + O(n^{\frac{1}{3}} \log n)$
2. $\lambda_1 \sim N((n-1)\mu + \nu + \frac{\sigma^2}{n}, 2\sigma^2)$[1]

*Proof.* The proof is derived by looking at moments of increasing order. An alternative approach, by Kahn and Szemeredi, relies on the Rayleigh quotient to understand the behavior of random matrices. □

**Theorem 3.** *(Broder & Shamir '87) : For almost all $d$-regular graphs, $\lambda_2 = O(d^{\frac{3}{4}})$.*

*Proof.* Let $G$ be a random $2d$-regular graph on $n$ vertices, generated by choosing $d$ random permutations, $\pi_1, \dots, \pi_d$, and defining an edge $(v, \pi_i(v))$ for every $v \in [n]$ and $i \in [d]$. Let $P$ be the transition matrix of the Markov chain defined by the graph, i.e. it's adjacency matrix divided by $2d$. Let $\mu_1 \geq \mu_2 \geq \dots \geq \mu_n$ be its eigenvalues, and $\rho = \max\{|\mu_2|, |\mu_n|\}$. Since $\mu_1 = 1$, and for any $k$, $\{\mu_i^k\}_{i=1}^n$ are the eigenvalues of $P^k$, we have $\rho^{2k} \leq tr(P^{2k}) - 1$, and therefore:

$$\mathbb{E}(\rho) \leq (\mathbb{E}(\rho^{2k}))^{\frac{1}{2k}} \leq (\mathbb{E}(tr(P^{2k})) - 1)^{\frac{1}{2k}}, \quad (1)$$

where the inequality on the left follows from Jansen's inequality.

Let $v$ be a vertex of $G$. A path in the graph that starts at $v$ is uniquely defined be a word $S$ over the alphabet $\{\pi_1, \pi_1^{-1}, \dots, \pi_d, \pi_d^{-1}\}$ (these permutations label the edges of the path). Let us generate the graph by going along a random path, and choosing $\pi_i(u)$ uniformly among the currently allowed values. In other words, say we are currently at a vertex $u$. We choose uniformly at random $i \in [d]$ and $\epsilon \in \{-1, 1\}$. If $\pi_i^\epsilon(u)$ is already defined, we move to that vertex. We call this a "forced move". Otherwise, we choose $\pi_i^\epsilon(u)$ uniformly at random among the values not yet taken by $\pi_i^\epsilon$. We call this a "free move".

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[1] $N(\mu, \sigma^2)$ refers to the Normal Distribution with expectation $\mu$ and variance $\sigma^2$.

Define $red(S)$ to be the reduction of $S$, that is, what remains of $S$ if we repeatedly remove all two consecutive letters in it of the form $\pi_i, \pi_i^{-1}$.

Since we want to bound $tr(P^{2k})$, we would like to bound the probability that $S$ induces a path that starts and ends in $v$. We divide this event into three:

1. $red(S)$ is the identity.
2. The path defined by $S$ has exactly one (simple) loop.
3. The path defined by $S$ has at least two loops.

We bound each of these in the following three lemmas:

**Lemma 4.** *Let $S$ be a word of length $2k$ generated as above, then:*

$$\Pr[red(S) = \emptyset] \leq (\frac{2}{d})^k$$

*Proof.* The idea is to count closed paths in the infinite $2d$-regular tree, using Catalan numbers. $\square$

**Lemma 5.** *Let $S'$ be a word of length $2k$ generated as above, and let $S = red(S')$. Denote $s = |S|$, and assume $s > 1$, then:*

$$\Pr[S \textit{ start at 1, has exactly one loop, and ends in 1}] \leq \frac{1}{n} + O(\frac{s}{n^2})$$

*Proof.* Assume w.l.o.g. that $S$ is a loop (otherwise, we argue for the loop, which is even shorter than $s$), and denote the vertices it visits $1 = v_0, \ldots, v_s = 1$. Since all these vertices are distinct, the choice made at $v_{s-1}$ is free. Therefore, the probability that at this point vertex $1$ is chosen is $\frac{1}{n-s} = \frac{1}{n} + O(\frac{s}{n^2})$ $\square$

**Lemma 6.** *Let $S$ be a word of length $2k$ generated as above,then:*

$$\Pr[S \textit{ has two loops}] \leq O(\frac{k^4}{n^2})$$

*Proof.* For $S$ to have two loops there have to be two "free choices" where we choose a vertex that was already visited before. The probability of choosing a vertex that was already chosen before, is at most $\frac{2k}{n}$. The probability of this happening at two specific steps, is at most $\frac{4k^2}{n^2}$. By the union bound, the probability that it happens at some two steps is $O(\frac{k^4}{n^2})$. $\square$

Note that Lemma 5 is not exactly what we need, since it deals with a reduced word. Broder and Shamir also bound the probability that when $|S| = 2k$, $|red(S)| = s$. With this, they show that the probability of the second case is bounded by $\frac{1}{n} + O(\frac{k2^k}{nd^k})$. This, together with the other two bounds yields:

$$\Pr[S \text{ starts at v and returns to v}] \leq (\frac{2}{d})^k + \frac{1}{n} + O(\frac{k2^k}{nd^k} + \frac{k^4}{n^2}) \tag{2}$$

Finally, we need to choose the $k$ that minimizes the RHS, so we roughly need $(\frac{2}{d})^k = \frac{k^4}{n^2}$, or $k \sim (2-o(1))\log_{d/2} n$. Putting this back in 1, we get $\mathbb{E}(\rho) \leq (\frac{2}{d})^{\frac{1}{4}}(1 + o(1))$. The proof is finished by showing that $\rho$ is concentrated around its mean by using martingales. $\square$

**Note:** Currently, the best result is by Joel Friedman, who showed $\lambda_2 \leq \sqrt{2d-1} + \epsilon$, for all $\epsilon > 0$.

**Note:** The random model of Broder & Shamir, that of choosing $\frac{d}{2}$ random matchings, does not induce a uniform distribution on $d$-regular graphs. However, this model is *contiguous* to the uniform model, that is, any graph property that occurs in one of them w.h.p., occurs w.h.p. in the other as well. For details see chapter 9 in "Random Graphs" by S. Janson, T. Luczak and A. Ruciński.
Another source of reference is Nick Wormald's survey, "Models of random regular graphs", which appears in "Surveys in Combinatorics", 1999, J.D. Lamb and D.A. Preece, eds.