Expander Graphs and their Applications*
Lecture 13: On eigenvalues of random graphs and the abundance of Ramanujan graphs

Notes taken by Danny Gutfreund

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Summary: In this lecture we will be looking at eigenvalue distributions of random graphs and matrices. Our starting point is the question: Is it true that almost every graph is Ramanujan?

First, we consider generalizations of this question that look at the distribution of eigenvalues in general (and not only the second eigenvalue). We state "Wigner's semicircle law", and give a partial proof to a version of this law (By McKay) for regular graphs.

In the second part of the lecture, we define lifts of graphs and extend the definition of Ramanujan graphs to general (non-regular) graphs. We then state some conjectures and results regarding the abundance of Ramanujan graphs (under this new definition).

1 The eigenvalue distribution of random matrices and regular graphs

Open problem 1. *Is it true that almost every d-regular graph is Ramanujan? More formally, is it true that,*

$$\lim_{n\to\infty} Pr(\lambda_2(G_n) \leq 2\sqrt{d-1}) = 1$$

Where G_n is a random d-regular graph of size n.

Friedman gave a positive answer upto an ϵ additive factor,

Theorem 2. *For every $\epsilon > 0$,*

$$\lim_{n\to\infty} Pr(\lambda_2(G_n) \leq 2\sqrt{d-1} + \epsilon) = 1$$

Where G_n is a random d-regular graph of size n.

Extending this question to random symmetric matrices, and the distribution of all the eigenvalues, we have the following theorem by Wigner, known as "Wigner's semicircle law".

Theorem 3. *Let A_n be a $n \times n$ symmetric matrix over $\mathbb{R}$, where a_{ij} $(i \neq j)$ are sampled independently from a distribution F, and a_{ii} from a distribution G.*
Let $\lambda_1(A_n) \geq \cdots \geq \lambda_n(A_n)$ be the eigenvalues of A_n.
Define the empiric distribution,

$$W_n(x) = \frac{1}{n}|\{i : \lambda_i(A_n) \leq x\}|$$

Let,

$$W(x) = \lim_{n\to\infty} W_n(2x\sigma\sqrt{n})$$

where $\sigma^2 = var(F) = var(G)$.
If it holds that $\forall k, \int |x|^k dF, \int |x|^k dG < \infty$, then $W(x)$ is continuous with density $\frac{2}{\pi}\sqrt{1-x^2}$ if $|x| \leq 1$, and 0 otherwise.

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Going back to eigenvalues of adjacency matrices of d-regular graphs, we have the following version of Wigner's semicircle law, proven by McKay.

Theorem 4. *Let G_n be an infinite sequence of d-regular graphs, such that, for all $k \geq 3$, $C_k(G_n) = o(|G_n|)$, where $C_k(G_n)$ is the number of length k cycles in G_n.*
Define,

$$F(G_n, x) = \frac{1}{|G_n|}|\{i : \lambda_i(G_n) \leq x\}|$$

Then for every x,

$$F(x) = \lim_{n\to\infty} F(G_n, x) = \int_{-2\sqrt{d-1}}^{x} \frac{d\sqrt{4(d-1) - z^2}}{2\pi(d^2 - z^2)} dz$$

The idea of the proof is that under the assumption that there are very few short cycles, the neighborhood of almost every node looks like a tree. So we can count the number of cycle-free paths from a node to itself. We state this formally in the following lemma.

Lemma 5. *Let G be a d-regular graph, and let v be a node such that there are no cycles in its r-neighborhood. Then the number of paths of length r that start and end in v is 0 if r is odd, and if $r = 2s$ it is,*

$$\psi(r) = \sum_{j=1}^{s} \binom{2s-j}{s} \frac{j}{2s-j} d^j (d-1)^{s-j}$$

Proof. Clearly, if the r-neighborhood of v does not contain any cycles, then every length r path that starts and ends in v is cycle-free, and hence r must be even.

Let $r = 2s$. Every length r path that starts and ends in v, defines a sequence, $0 = \delta_0, \delta_1 \cdots \delta_r = 0$. Where δ_i is the distance from v after we did i steps on the path. Clearly, for every i, $\delta_i \geq 0$, and $|\delta_i - \delta_{i-1}| = 1$. We would like to count the number of such sequences in which exactly j out of $\delta_0 \cdots \delta_r$ are 0. This is a simple generalization of Catalan numbers and the answer is,

$$\binom{2s-j}{s} \frac{j}{2s-j}$$

We know that a path that defines such a sequence visits v exactly j times. Each time it leaves v it has exactly d possibilities for the next step. This gives the term d^j. In steps that go away from v, i.e. when $\delta_{i+1} - \delta_i = 1$, we have $d-1$ different nodes that we can continue too (we cannot backtrack to the previous node). and for steps that go toward v we have no choice because we have to go back on the same edge that we used before. There are $s - j$ steps of each type, altogether we have $(d-1)^{s-j}$ possibilities for such steps. □

If $C_k(G_n) = o(|G_n|)$ (for every $k \geq 3$), then for every constant r, almost every node has cycle-free r-neighborhood. Let $P_r(G_n)$ be the number of simple paths of length r from a node to itself in G_n. Then from Lemma 5 we conclude that,

$$\lim_{n\to\infty} \frac{P_r(G_n)}{G_n} = \psi(r)$$

Therefore, the function $F(x)$ must satisfy $\int x^r dF = \psi(r)$, for every r. In order to finish the proof of the theorem we need some inverse transformation, that calculates $F(x)$ out of its moments. This is achieved using the Chebyshev polynomials, but we do not include the details in this lecture note.

2 Random lifts and general Ramanujan graphs

We start by defining the notion of coverings and lifts.

Definition 6. Let G and H be two graphs. We say that a function $f : V(H) \to V(G)$ is a covering, if for every $v \in V(H)$, the restriction of f to the set of neighbors of v, $\Gamma_H(v)$, is one to one and onto $\Gamma_G(f(v))$.
If a covering function from H to G exists, we say that H lifts G.

Example 7. The zig-zag product of G and H (where H is the smaller of the two) lifts H^2.

Remark 8. *If G is connected then every covering function on G has a covering number n, such that for every $v \in V(G)$, $|f^{-1}(v)| = n$, and for every $e \in E(G)$, $|f^{-1}(e)| = n$.*

Definition 9. Let f be a covering function on G. For a node $v \in V(G)$, we say that $f^{-1}(v)$ is the fiber of v.

Let G be a connected graph. We denote by $L_n(G)$ the set of graphs that are lifts of G with covering number n. We would like to characterize the members of $L_n(G)$. If $H \in L_n(G)$ then $V(H) = V(G) \times [n]$. That is, for every $v \in V(G)$, the nodes $(v, 1), \cdots, (v, n)$ in $V(H)$ are the fiber of v. Next we define the edges of H, for every edge $(v, u) \in E(G)$ we take some permutation $\pi \in S_n$ and define the following edges in H, $((v, i), (u, \pi(i)))$ (for every $i \in [n]$). Thus every choice of permutations defines a member in $L_n(G)$. This also gives us a way to sample random elements from this set.

What can we say about the eigenvalues of lifts of G? We know that the eigenvalues of G are also the eigenvalues of its lifts. To see this, let $h : V(G) \to \mathbb{R}$ be an eigenfunction of G with eigenvalue λ, and let H cover G with the map $f : V(H) \to V(G)$, then $h \circ f$ is an eigenfunction of H with the same eigenvalue. Thus if H lifts G, we can talk about its *old* eigenvalues, i.e. those that were inherited from G, and its *new* eigenvalues, which are the rest of the eigenvalues.

Definition 10. The universal covering space of a graph G is a graph that lifts all the lifts of G.

Example 11. The infinite d-regular tree is the universal covering space of every d-regular graph.

Grienberg and Lubotzky gave the following definition which extends the definition of Ramanujan graphs to general (non-regular) graphs.

Definition 12. We say that a graph G is Ramanujan if the absolute value of every eigenvalue of G except λ_0 is at most the spectral radius [1] of the universal covering space of G.

Conjecture 13. *For every graphs G, if we lift it high enough then almost surely we will get a Ramanujan graph.*

Lubotzky and Nagnibeda falsified this conjecture by constructing an infinite tree T, that covers infinite number of finite graphs such that none of them is Ramanujan.

Friedman showed that the construction of Lubotzky and Nagnibeda, in fact, constructs a single graph G^* that is covered by all the graphs that T covers and has large second eigenvalue (i.e. larger than the spectral radius of T). Since all the other graphs inherit the eigenvalues of G^*, none of them can be Ramanujan. He then rephrased Conjecture 13 as follows,

Conjecture 14. *For every graph G, if we lift it high enough to a graph H, then the* new *eigenvalues of H will almost surely be at most the spectral radius of the universal covering of H.*

To support his conjecture, Friedman proved the following theorem.

Theorem 15. *Let G be a graph with a largest eigenvalue λ_0, and let ρ be the spectral radius of its universal covering space. Then in almost every (high enough) lift of G, every new eigenvalue μ satisfies,*

$$\mu \leq \sqrt{\lambda_0 \rho} + o(1)$$

This is a generalization of the following result by Broder and Shamir.

Theorem 16. *For almost every d-regular graph, the second eigenvalue is at most $O(d^{\frac{3}{4}})$.*

To see that Theorem 15 generalizes 16, note that for d-regular graphs, $\lambda_0 = d$ and $\rho = \sqrt{d}$.

[1] We refer the reader to lecture 5. There we defined the spectrum $\sigma(A_T)$ of the adjacency matrix of an infinite tree T. The spectral radius is the maximal (absolute) value in $\sigma(A_T)$.