Expander Graphs and their Applications*
Lecture 7: The Margulis construction

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Summary: We define an explicit family of 8-regular graphs on the torus $\mathbb{Z}_n \times \mathbb{Z}_n$, and prove that this is a family of expander graphs.

Construction 1. Let

$$T_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix},\, T_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix},\, e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},\, e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and define the following 8-regular graph G=(V,E) on the vertex set $V = \mathbb{Z}_n \times \mathbb{Z}_n$. Each vertex $v = (x, y)$ is adjacent to the four vertices

$$T_1 v,\, T_2 v,\, T_1 v + e_1,\, T_2 v + e_2,$$

where all the calculations are performed mod n. The other four neighbours of v are obtained by the four inverse transformations. (Note that this is an 8-regular undirected graph, that may have multiple edges and self loops.)

Theorem 2. $\lambda_2(G) \leq 5\sqrt{2} < 8$

We have already seen that if G is a d-regular graph of size n and eigenvalues (of its adjacency matrix) $\lambda_1 \geq \lambda_2 \ldots \geq \lambda_n$, then $\lambda_1 = d$, then its Cheeger constant satisfies $h(G) \geq (d - \lambda_2)/2$. Therefore, a lower bound on the gap between λ_2 and d that is independent of n implies that this is a family of expanding graphs. Margulis proved a similar result in 1973 but couldn't give an explicit lower bound on the gap. Galil and Gabber (1981) used continuous harmonic analysis to derive a lowerbound on the gap. Boppana simplified the proof, and Jimbo, Marouka (1985) improved it furthermore using discrete Fourier transform.

As we have seen before, by the Rayleigh quotient Theorem,

$$\begin{aligned} \max\{|\lambda_2|, |\lambda_n|\} &= \max\{| < Af, f > | :< f, v_1 >= 0, ||f|| = 1\} \\ &= 2\max\{| \sum_{(i,j)\in E} \overline{f(i)} f(j)| :< f, v_1 >= 0, ||f|| = 1\}, \end{aligned}$$

where $v_1 = \overline{1}/\sqrt{n}$ is the eigenvector corresponding to λ_1, and the maxima are taken over all $f : \mathbb{Z}_n^2 \to \mathbb{C}$. It follows, that it is sufficient to prove that for every complex function $f : \mathbb{Z}_n^2 \to \mathbb{C}$ satisfying $\sum_x f(x) = 0$,

$$| \sum_{(x,y)\in E} \overline{f(x)} f(y)| \leq \frac{5\sqrt{2}}{2} \sum |f(x)|^2.$$

By the definition of our graph, this is equivalent to:

Theorem 3. *For any $f : \mathbb{Z}_n^2 \to \mathbb{C}$ satisfying $\sum_\vartheta f(\vartheta) = 0$, the following inequality holds:*

$$| \sum_{\vartheta\in\mathbb{Z}_n^2} \overline{f(\vartheta)} \big(f(T_1\vartheta) + f(T_1\vartheta + e_1) + f(T_2\vartheta) + f(T_2\vartheta + e_2)\big)| \leq \frac{5\sqrt{2}}{2} \sum |f(x)|^2 \tag{1}$$

* Lecture Notes for a course given by Nati Linial and Avi Wigderson at the Hebrew University, Israel.

Discrete Fourier transform It is hard to use the condition $\sum_x f(x) = 0$ so we will move to a new space using the discrete Fourier transform.

A character of an abelian group G is a homomorphism $\chi : G \to \mathbb{C}$, mapping addition in G to multiplication in $\mathbb{C}$. It can be seens, that the characters of $\mathbb{Z}_n^k$ are: $\chi_b : \mathbb{Z}_n^k \to \mathbb{C}$ for $b \in \mathbb{Z}_n^k$, where $\chi_b(a) = \omega^{<a,b>}$. Here, ω is the nth root of the unity ($\omega = e^{2\pi i/n}$), and for any $a, b \in \mathbb{Z}_n^k$ their inner product is $<a, b>= \sum_{j=1}^k \overline{a_i} b_i$.

Since the characters χ_b are an orthonormal basis, we can express any $f : \mathbb{Z}_n^k \to \mathbb{C}$ as $f = \sum_b \widehat{f}(b)\chi_b$, where

$$\widehat{f}(a) =< f, \chi_b >= \frac{1}{n^{k/2}} \sum_b \overline{f(b)} \cdot \chi_b(a) = \frac{1}{n^{k/2}} \sum_b \overline{f(b)} \cdot \omega^{<a,b>}.$$

$\widehat{f} : \mathbb{Z}_n^k \to \mathbb{C}$ is called the Discrete Fourier Transform - (DFT) of $f : \mathbb{Z}_n^k \to \mathbb{C}$.

To prove Theorem 3, we express the condition of equation (1) using the Fourier coefficients of f. The condition $\sum_\vartheta f(\vartheta) = 0$ is equivalent to $\widehat{f}(0,0) = 0$. Using identities 2,4 in appendix 1, one can easily prove that Theorem 3 reduces to:

Theorem 4. *Any function $F : \mathbb{Z}_n^2 \setminus (0,0) \to \mathbb{C}$ must satisfy*

$$|\sum_{\vartheta=(\vartheta_1,\vartheta_2)\in\mathbb{Z}_n^2\setminus(0,0)} \overline{F(\vartheta)} \cdot [F(T_2^{-1}\vartheta)(1+\omega^{-\vartheta_1}) + F(T_1^{-1}\vartheta)(1+\omega^{-\vartheta_2})]| \leq \frac{5\sqrt{2}}{2} \sum_{\vartheta\in\mathbb{Z}_n^2\setminus(0,0)} |F(\vartheta)|^2.$$

Let $G = |F|$. Then G is a real function $G : \mathbb{Z}_n^2 - (0,0) \to \mathbb{R}$. Moving the absolute value inside the summation, and using the equality

$$|1 + \omega^{-a}| = 2|\cos(\frac{\pi a}{n})|,$$

it follows that it is sufficient to prove:

Theorem 5. *Any function $G : \mathbb{Z}_n^2 - (0,0) \to \mathbb{R}$, must satisfy*

$$\sum_\vartheta G(\vartheta) \cdot [G(T_2^{-1}\vartheta)|\cos\frac{\pi\vartheta_1}{n}| + G(T_1^{-1}\vartheta)|\cos\frac{\pi\vartheta_2}{n}|] \leq \frac{5\sqrt{2}}{4} \sum G^2(\vartheta).$$

We would like to convert the LHS into a sum of squares that will match the RHS. To do this we use the following inequality, valid for any real a, b, γ

$$2ab \leq \gamma a^2 + \frac{1}{\gamma} b^2.$$

Let $\gamma : (\mathbb{Z}_n^2)^2 \to \mathbb{R}$ be a function satisfying for all $x, y \in \mathbb{Z}_n^2$

$$\gamma(x, y) \cdot \gamma(y, x) = 1, \tag{2}$$

and in addition, that for all $\vartheta = (\vartheta_1, \vartheta_2) \in \mathbb{Z}_n^2 \setminus (0,0)$

$$|\cos\frac{\pi\vartheta_1}{n}| \cdot [\gamma(\vartheta, T_2\vartheta) + \gamma(\vartheta, T_2^{-1}\vartheta)] + |\cos\frac{\pi\vartheta_2}{n}| \cdot [\gamma(\vartheta, T_1\vartheta) + \gamma(\vartheta, T_1^{-1}\vartheta)] \leq \frac{5\sqrt{2}}{2}. \tag{3}$$

Since for any $p, q \in \mathbb{Z}_n^2 \setminus (0,0)$

$$2G(p)G(q) \leq \gamma(p,q)G^2(p) + \gamma(q,p)G^2(q),$$

it follows that

$$\begin{aligned} 2 \cdot LHS \ \leq \ \textstyle\sum_\vartheta \ & |\cos\frac{\pi\vartheta_1}{n}| \cdot [\gamma(\vartheta, T_2^{-1}\vartheta)G^2(\vartheta) + \gamma(T_2^{-1}\vartheta, \vartheta)G^2(T_2^{-1}\vartheta)] \\ + \ & |\cos\frac{\pi\vartheta_2}{n}| \cdot [\gamma(\vartheta, T_1^{-1}\vartheta)G^2(\vartheta) + \gamma(T_1^{-1}\vartheta, \vartheta)G^2(T_1^{-1}\vartheta)]. \end{aligned}$$

Since T_1 doesn't change ϑ_2, and T_2 doesn't change ϑ_1, it follows that:

$$\begin{aligned} 2 \cdot LHS &\leq \sum_{\vartheta} G^2(\vartheta) \cdot |\cos\frac{\pi\vartheta_1}{n}| \cdot [\gamma(\vartheta, T_2\vartheta) + \gamma(\vartheta, T_2^{-1}\vartheta)] + G^2(\vartheta) \cdot |\cos\frac{\pi\vartheta_2}{n}| \cdot [\gamma(\vartheta, T_1\vartheta) + \gamma(\vartheta, T_1^{-1}\vartheta)] \\ &\leq \frac{5\sqrt{2}}{2} \sum_{\vartheta} G^2(\vartheta). \end{aligned}$$

Therefore, if we just find a function γ satisfying the requirements (2) and (3), Theorem 2 would follow, and we are done. To define γ, we first define a partial order on $\mathbb{Z}_n^2$. So, let

$$a(x) = \begin{cases} x & \text{if } \frac{n}{2} \geq x \geq 0\ , \\ n - x & \text{if } n \geq x \geq \frac{n}{2}. \end{cases}$$

(Notice that $a(x)$ is invariant under mod n, i.e. $a(x \mod n) = a(x)$.) Then we say that $(\vartheta_1, \vartheta_2) > (\vartheta_1', \vartheta_2')$ if $a(\vartheta_1) \geq a(\vartheta_1')$ and $a(\vartheta_2) \geq a(\vartheta_2')$ and at least one of the inqualities is strong. That is the distance to the X axis and/or to the Y axis is bigger.

The definition of γ is:

$$\gamma((\vartheta_1, \vartheta_2), (\vartheta_1', \vartheta_2')) = \begin{cases} \alpha & \text{if }, (\vartheta_1, \vartheta_2) > (\vartheta_1', \vartheta_2') \\ \frac{1}{\alpha} & \text{if }, (\vartheta_1, \vartheta_2) < (\vartheta_1', \vartheta_2') \\ 1 & \text{otherwise.} \end{cases}$$

This definition of γ obviously satisfies (2). We will show that for $\alpha = \frac{5}{4}$, also (3) is satisfied for any $\vartheta \in \mathbb{Z}_n^2 \backslash (0, 0)$. We define the diamond to be the set of all $\vartheta = (\vartheta_1, \vartheta_2) \in \mathbb{Z}_n^2 \setminus (0, 0)$, satisfying

$$a(\vartheta_1) + a(\vartheta_2) \leq \frac{n}{2}.$$

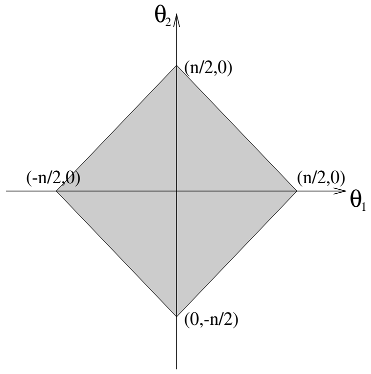

Figure 1: The diamond

We analyze the case where ϑ is outside or inside the diamond separately.

- **Outside the diamond**

  (3) will follow from the fact that the cosines are small. So, it is sufficient to prove that

$$2\alpha \cdot (|\cos\frac{\pi\vartheta_1}{n}| + |\cos\frac{\pi\vartheta_2}{n}|) \leq \frac{5\sqrt{2}}{2},$$

  or that

$$|\cos\frac{\pi\vartheta_1}{n}| + |\cos\frac{\pi\vartheta_2}{n}| \leq \sqrt{2}. \tag{4}$$

We can assume w.l.o.g. that we are in the first quadrant. Since $\cos\frac{\pi\vartheta_2}{n}$ is decreasing, for any given value of ϑ_1, the LHS of (4) is maximized, when $\vartheta_2 = n/2 - \vartheta_1$. Therefore, using the convexity of $\cos$ we get that

$$\cos\frac{\pi\vartheta_1}{n} + \cos\frac{\pi\vartheta_2}{n} \leq \cos\frac{\pi\vartheta_1}{n} + \cos\frac{\pi(\frac{n}{2} - \vartheta_1)}{n} \leq 2\cos\frac{\pi}{4} = \sqrt{2}.$$

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- **Inside the diamond**

  In this case, we just bound the cosines by 1, and would like to prove that

  $$\gamma(\vartheta, T_1\vartheta) + \gamma(\vartheta, T_1^{-1}\vartheta) + \gamma(\vartheta, T_2\vartheta) + \gamma(\vartheta, T_2^{-1}\vartheta) \leq \frac{5\sqrt{2}}{2}. \quad (5)$$

  It is not difficult to verify that for every ϑ satisfying $a(\vartheta_1) + a(\vartheta_2) < \frac{n}{2}$, one of the following two cases must hold:

  1. Three of the four points $T_1\vartheta$, $T_2\vartheta$, $T_1^{-1}\vartheta$ and $T_2^{-1}\vartheta$ are $> \vartheta$ and one is $< \vartheta$.
  2. Two of the four points $T_1\vartheta$, $T_2\vartheta$, $T_1^{-1}\vartheta$ and $T_2^{-1}\vartheta$ are $> \vartheta$ and two are incomparable with ϑ.

  In the first case, the LHS of (5) is $\leq \frac{3}{\alpha} + \alpha$, while in the second case it is $\leq \frac{2}{\alpha} + 2$. Substituting $\alpha = 5/4$, this gives a upper bound of 3.65 or the LHS of (3). This is not as good as $5\sqrt{2}/2 = 3.53\ldots$, but it does prove that the graphs are a family of good expanders.

Appendix 1. *Using the same definitions as before the following properties of the discrete Fourier transform can by easily concluded from the definitions:*

1. $\sum_a f(a) = 0 \Leftrightarrow \widehat{f}(0) = 0$
2. *The dot product of two functions remains invariant under the transform i.e for any f, g*

   $$\sum_a \overline{f(a)}g(a) = \sum_a \overline{\widehat{f}(a)}\widehat{g}(a).$$

3. *Parseval's identity: A special case of 2, where $f = g$*

   $$\sum_a |f(a)|^2 = \sum_a |\widehat{f}(a)|^2.$$

4. *The shift property : If A is a non-singular $k \times k$ matrix with entries over $\mathbb{Z}_n$, $b \in \mathbb{Z}_n^k$ and $g(x) = f(Ax + b)$ then*

   $$\widehat{g}(y) = \omega^{-<A^{-1}b,y>}\widehat{f}((A^{-1})^t y).$$

5. *The inverse formula :*

   $$f(a) = \frac{1}{n^{k/2}}\sum_b \widehat{f}(a)\omega^{-<a,b>}.$$