Expander Graphs and their Applications*
Lecture 6: Upper Bound on Spectral Gap

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Summary: This lecture presents a lower bound for λ_1, the second-largest eigenvalue of a d-regular graph: $\lambda_1 \geq 2\sqrt{(d-1)}(1-\frac{c}{\Delta^2})$, which is related to the graph's diameter Δ. As the diameter grows, so does the lower bound for λ_1. This can also be viewed as an upper bound on the graph's spectral gap.

1 Reminder to previous lecture

In the previous lecture we discussed the infinite d-regular tree and the spectrum of its (infinite) adjacency matrix A. We defined the spectrum $\sigma(A)$ as follows:

$$\sigma(A) := \{\lambda | (\lambda I - A) \; is\ not\ invertible\ \}$$

And we showed that for the d-regular infinite tree:

$$\sigma(A) = [\, -2\sqrt{d-1}\;,\; 2\sqrt{d-1}\,]$$

In this lecture we will use the finite (k-tall) d-regular tree to prove a lower bound on λ_1 for general graphs.

2 Lower bound on λ_1

We show a lower bound on λ_1 for general d-regular graphs: [1]

Theorem 1. [2] *There exists a constant c s.t. for any d-regular graph G of size n and diameter Δ:*

$$\lambda_1 \geq 2\sqrt{(d-1)}(1-\frac{c}{\Delta^2})$$

Where λ_1 is the second largest eigenvalue of G.

Notes:

- It is easy to show that in the above graph $Diam(G) > \Omega(\log_{d-1} n)$, and hence it follows that for any fixed $d > 2$:

$$\lambda_1 \geq 2\sqrt{(d-1)}(1-O(\frac{1}{\log^2 n})).$$

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[1] The original statement of the lower bound is due to N. Alon and R. Boppana, and appears in A. Nilli *On the second eigenvalue of a graph* Discrete Math., 91(2):207-210, 1991

[2] This stronger statement and proof is taken from J. Friedman *Some geometric aspects of graphs and their eigenfunctions* Duke Math. J. 69(3):487-525, 1993.

- In the case that $n = d + 1$ (G is the n-clique) the eigenvalues of G are $\{ n-1, -1, \ldots, -1 \}$, since if A is the adjacency matrix for G, then $A + I = J$ and J's eigenvalues are, naturally, $\{ n, 0, \ldots, 0\}$. While at first this may seem to be a counter-example to our theorem, note that the theorem deals with the case where d is fixed and n, Δ are going to infinity.
- Since $\lambda_1 = \max_{x \perp \vec{1}} \{ \frac{xAx^T}{||x||^2} \}$ we expect a proof for the above bound might use a specific "test function" (=eigenvector), e.g. find a vector f s.t. $\sum_{x \in V(G)} f(x) = 0$ and $\frac{fAf^T}{||f||^2} \geq 2\sqrt{d-1}(1 - \frac{c}{\Delta^2})$.

Proof Sketch: Taking two nodes s, t with $d(s,t) = \Delta$, we build a spherical function f that will be positive for the nodes within a distance of $k = \lfloor \frac{\Delta}{2} \rfloor - 1$ from s, and negative on the nodes that are within a distance k from t. The values of f will be derived from the spherical function g with maximal eigenvalue μ for the d-regular tree of height k, treating s and t as roots of (separate) k-tall trees. We will show that for the positive part of f we have $Af \geq \mu f$, and likewise for the negative part of f we have $Af \leq -\mu f$, so that $fAf^t \geq \mu f f^t = \mu ||f||^2$, giving $\frac{fAf^T}{||f||^2} \geq \mu$. Finally, the positive and negative parts of f will be normalized to ensure $\sum f(x) = 0$, letting μ apply as a lower bound for λ_1. □

Proof. Set $k = \lfloor \frac{\Delta}{2} \rfloor - 1$. Select two nodes $s, t \in G$ with distance $d(s,t) = \Delta$. For all $0 \leq i \leq k$ define:

$$S_i := \{v | d(s, v) = i\}$$

and

$$T_i := \{v | d(t, v) = i\}$$

and in addition define

$$Q := V(G) \setminus (\bigcup_{0 \leq i \leq k} S_i \cup T_i)$$

> **Note:** This is simply a breadth-first-search dividing the graph into layers according to the distance from s and t. Q represents the "middle ground" with at least 1 layer of nodes. There are, of course, no edges between any S_i and any T_i.

Denote T to be the finite tree of height k, and mark A_T to be its adjacency matrix.

Claim 2. *There is a single spherical function $g : [0, \ldots, k+1] \to \mathbb{R}$ on T that satisfies:*

$$g(k+1) = 0$$

(we extend g's domain in include $k+1$ even though it is not part of A_T), and

$$A_T g = \mu g$$

(g is an eigenfunction of A_T with eigenvalue μ), where μ is the maximal eigenvalue of A_T.

This claim can be proved using the same technique as shown in the previous lecture dealing with the infinite d-tree. We will not prove this claim here, but will show a specific function g that displays these properties.

Definition 3. Define $f : V(G) \to \mathbb{R}$ as follows:

$$f(v) = \begin{cases} c_1 g(i) & \exists i, v \in S_i \\ -c_2 g(i) & \exists i, v \in T_i \\ 0 & otherwise \end{cases}$$

where $c_1, c_2 \geq 0$.

Since f is spherical, it will be convenient to define $F : [0, ..., k] \to \mathbb{R}$ with $f(v) = F(i) \iff v \in S_i$. We will next show that f gives us the desired properties:

Lemma 4. *If g above is non-negative and monotonically non-increasing, then:*

$$v \in S_i => (Af)(v) \geq \mu f(v)$$

and

$$v \in T_i => (Af)(v) \leq \mu f(v)$$

Proof. Let $v \in S_i$ for some i. G is d-regular, so v has $1 \leq p \leq d$ neighbors in level $i-1$, another $0 \leq q \leq d-p-1$ neighbors in the same level i, and the remaining $d-p-q$ neighbors in level $i+1$, giving:

$$\begin{aligned}(Af)(v) &= pF(i-1) + qF(i) + (d-p-q)F(i+1) \\ &= pc_1g(i-1) + qc_1g(i) + (d-p-q)c_1g(i+1)\end{aligned}$$

but for g and the matrix A_T (of the k-tall d-tree), we know that for a node u of level i:

$$\mu g(i) = (A_T g)(i) = g(i-1) + (d-1)g(i+1)$$

since each node has exactly one neighbor in the previous level, and $d-1$ neighbors in the next level. As g is non-negative and non-increasing we get:

$$\begin{aligned}(Af)(v) &= c_1[pg(i-1) + qg(i) + (d-p-q)g(i+1)] \\ &\geq c_1[g(i-1) + (d-1)g(i+1)] \\ &= c_1(A_T g)(i) = c_1 \mu g(i) \\ &= \mu f(v)\end{aligned}$$

The same argument is used for $v \in T_i$, with the approriate redefinition of the function F, and using c_2.

□

Corollary 5. $\frac{fAf^T}{||f||^2} \geq \mu$ *and* $f \perp \vec{1}$.

Proof. The previous lemma gives $|(Af)(v)| \geq |\mu f(v)|$ for $v \in V(G) \setminus Q$. For $v \in Q$, note that $f(v) = 0$, in which case $|(Af)(v)| \geq |\mu f(v)|$ is trivial. From this follows:

$$\begin{aligned}fAf^T &= <f, Af> = \sum_{v \in V(G)} f(v)(Af)(v) \\ &= \sum_{\exists i, v \in S_i} f(v)(Af)(v) + \sum_{\exists i, v \in T_i} f(v)(Af)(v) + \sum_{v \in Q} f(v)(Af)(v) \\ &\geq \sum_{\exists i, v \in S_i} f(v)\mu f(v) + \sum_{\exists i, v \in T_i} f(v)\mu f(v) \\ &= \mu f f^T\end{aligned}$$

since for $v \in T_i$ $f(v) \geq 0$ and for $v \in Q$ $f(v) = 0$. From this we get:

$$fAf^T \geq \mu f f^T => \frac{fAf^T}{||f||^2} \geq \mu$$

Next, we show that $f \perp \vec{1}$, (namely: $\sum f(v) = 0$). This is easily achieved by selecting c_1 and c_2 such that

$$\sum_{v \in S_i} f(v) = -\sum_{v \in T_i} f(v)$$

□

Definition 6. Finally, we will find the spherical function g that is non negative, non-increasing, that satisfies $g(k+1) = 0$, and that has the maximal eigenvalue μ. We'll also show that this μ yields the desired bound:

$$g(i) := (d-1)^{-i/2}\sin(\theta(k+1-i))$$

where $\theta = \frac{\pi}{2k+2}$.

It is easy to check that g is non-negative, non increasing, and that $g(k+1) = 0$. We now show that $A_T g = \mu g$ with $\mu = 2\sqrt{d-1}\cos(\theta)$. For all j:

$$\begin{aligned}
(A_T g)(j) &= 1g(j-1) + (d-1)g(j+1) \\
&= (d-1)^{-(j-1)/2}\sin(\theta(k+1-(j-1))) + (d-1)(d-1)^{-(j+1)/2}\sin(\theta(k+1-(j+1))) \\
&= (d-1)^{(-j+1)/2}\sin(\theta(k+2-j)) + (d-1)^{(-j+1)/2}\sin(\theta(k-j) \\
&= \sqrt{d-1}(d-1)^{-j/2}(\sin(\theta(k+2-j)) + \sin(\theta(k-j)))
\end{aligned}$$

And since $\sin(\alpha) + \sin(\beta) = 2\sin(\frac{\alpha+\beta}{2})\cos(\frac{\alpha-\beta}{2})$, we get for all j:

$$\begin{aligned}
(A_T g)(j) &= 2\sqrt{d-1}\cos(\theta)(d-1)^{-j/2}\sin(\theta(k+1-j)) \\
&= 2\sqrt{d-1}\cos(\theta)g(j)
\end{aligned}$$

To arrive at our bound, we need to show that $\cos(\theta) \geq (1 - \frac{c}{\Delta^2})$. Recall:

$$\theta = \frac{\pi}{2k+2} \tilde{=} \frac{\pi}{\Delta}$$

since $k = \lfloor \frac{\Delta}{2} \rfloor - 1$, and use the Taylor expansion for the fucntion $\cos(x)$:

$$\cos(x) = 1 - \frac{x^2}{2} + o(x^2)$$

to get:

$$\cos(\theta) \geq 1 - \frac{\pi^2}{2\Delta^2} = (1 - \frac{c}{\Delta^2})$$

with $c \approx \frac{\pi^2}{2}$. □