Expander Graphs and their Applications*
Lecture 5: Expander graphs have a large spectral gap

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Summary: In the previous lecture we started proving upper and lower bounds on $h(G)$, defined as $h(G) = \min_{|S|\leq\frac{n}{2}}(\frac{e(S,S^C)}{|S|})$. We proved a lower bound of $\frac{d-\lambda}{2} \leq h(G)$. In this lecture we will prove an upper bound of $h(G) \leq \sqrt{2d(d-\lambda)}$. We will also discuss the d-regular infinite tree, which is the optimal expander, and show that its spectrum is $[-2\sqrt{d-1}, 2\sqrt{d-1}]$.

1 Comments about the previous lecture

Given a graph G, we choose an arbitrary orientation for its edges, and define the matrix $M_{n\times m}$ where $n = |V(G)|$ and $m = |E(G)|$. The entries M_{ue} for $u = 1\ldots n$ and $e = 1\ldots m$, are defined as follows:

$$M_{ue} = \begin{cases} 1 & e = (u \to v) \\ -1 & e = (v \to u) \\ 0 & otherwise \end{cases}$$

The Laplacian of G is defined as the matrix $L = M \cdot M^T$. Then for any $f : V \to \mathbb{R}$, we have

$$fLf^T = fMM^Tf^T = < fM, fM > = \| fM \|^2$$

where $\|\cdot\|$ is the l_2 norm. Furthermore, $(fM)_{e=(x\to y)} = f(x) - f(y)$, and therefore we can write

$$\| fM \|^2 = \sum_{(x,y)\in E}(f(x)-f(y))^2$$

Another comment, is about the variational description of the eigenvalues of a matrix, which is

$$\lambda_k = \max_{x\perp x_1,\ldots,x_{k-1}}\left(\frac{xAx^T}{\| x \|^2}\right)$$

This can be re-written in the following way:

$$\lambda_k = \min_{F, dim(F)=n-k+1}\ \max_{x\in F}\left(\frac{xAx^T}{\| x \|^2}\right)$$

2 An upper bound on $h(G)$

Theorem 1. *For any connected graph G, define $h(G) = \min_{|S|\leq\frac{n}{2}}(\frac{e(S,S^C)}{|S|})$. Then*

$$h(G) \leq \sqrt{2d(d-\lambda)}$$

* Lecture Notes for a course given by Nati Linial and Avi Wigderson at the Hebrew University, Israel.

Proof. **Definition 2.** Given a function $f : V \to \mathbb{R}$, define

$$B_f = \sum_{(x,y)\in E} |f^2(x) - f^2(y)|$$

Definition 3. Let $\beta_0 < \beta_1 < \cdots < \beta_r$ be the different values achieved by f over V. Then we define L_i for $i = 1, \ldots, r$ as follows:

$$L_i = \{x \in V | f(x) \geq \beta_i\}$$

This definition leads to $L_0 \supseteq L_1 \supseteq \cdots \supseteq L_r$.

We will use the three following claims.

Claim 4. $B_f = \sum_{i=1}^{r} e(L_i, L_i^C)(\beta_i^2 - \beta_{i-1}^2)$.

Proof. For $(x, y) \in E$, if $f(x) = \beta_p > \beta_q = f(y)$, (x, y)'s contribution to B_f is

$$(\beta_p^2 - \beta_q^2) = (\beta_p^2 - \beta_{p-1}^2) + (\beta_{p-1}^2 - \beta_{p-2}^2) + \cdots + (\beta_{q+1}^2 - \beta_q^2)$$

When we sum over E, $(\beta_i^2 - \beta_{i-1}^2)$ appears once for every edge (x, y) such that $f(x) = \beta_p > \beta_q = f(y)$ and $p \geq i > q$. In other words, $(\beta_i^2 - \beta_{i-1}^2)$ appears exactly once for every edge (x, y) such that $x \in L_i$ and $y \notin L_i$, and in total $e(L_i, L_i^C)$ times, which gives us the desired result. □

Claim 5. $B_f \leq \sqrt{2d} \cdot \| fM \| \cdot \| f \|$.

Proof. Using the Cauchy-Schwartz inequality, we have

$$B_f = \sum_E |f^2(x) - f^2(y)| = \sum_E |f(x) + f(y)| \cdot |f(x) - f(y)| \leq$$

$$\leq \sqrt{\sum_E (f(x) + f(y))^2} \sqrt{\sum_E (f(x) - f(y))^2}$$

We know that

$$\sqrt{\sum_E (f(x) - f(y))^2} = \| fM \|$$

and evaluating the second term in the product gives us

$$\sqrt{\sum_E (f(x) + f(y))^2} \leq \sqrt{2 \sum_E (f^2(x) + f^2(y))} = \sqrt{2d \sum_V f^2(x)} = \sqrt{2d} \cdot \| f \|$$

from which we can conclude the inequality in the claim. □

Claim 6. *If $f \geq 0$ and $|Supp(f)| \leq \frac{n}{2}$, where $Supp(f)$ is the subset of V where $f(x) \neq 0$, then*

$$B_f \geq h(G) \| f \|^2$$

Proof. Since f equals zero on more than half of the coordinates, and is positive on the rest, $\beta_0 = 0$, and for every $i \geq 1$ $|L_i| \leq \frac{n}{2}$. Therefore $\frac{e(L_i, L_i^C)}{|L_i|} \geq h(G)$. Plugging this inequality into claim 4 yields

$$B_f = \sum_{i=1}^{r} e(L_i, L_i^C)(\beta_i^2 - \beta_{i-1}^2) \geq h(G) \sum_{i=1}^{r} |L_i|(\beta_i^2 - \beta_{i-1}^2) =$$

$$= h(G) \sum_{i=1}^{r} \beta_i^2 (|L_i| - |L_{i+1}|) = h(G) \sum_{i=1}^{r} \beta_i^2 \cdot |\{x | f(x) = \beta_i\}| = h(G) \| f \|^2$$

□

For each eigenvalue λ_i of A, $d - \lambda_i$ is an eigenvalue of the Laplacian, $L = dI - A$. Let g be an eigenvector of L (and A) with eigenvalue $d - \lambda$, where λ is the second largest eigenvalue of A. Define $f = g^+$, i.e. equal to g where g is positive, and zero elsewhere. Without loss of generality, $|Supp(f)| \leq \frac{n}{2}$, since otherwise we would look at $-g$ which is also an eigenvector with the same eigenvalue. Define $V^+ = Supp(f)$. Then for every $x \in V^+$ we can write

$$(Lf)(x) = df(x) - \sum_{y \in V} a_{xy} f(y) = dg(x) - \sum_{y \in V^+} a_{xy} g(y) \leq$$

$$\leq dg(x) - \sum_{y \in V} a_{xy} g(y) = (Lg)(x) = (d - \lambda) g(x)$$

Since $f(x) = 0$ for any $x \notin V^+$, we can write

$$\| fM \|^2 = fLf^T = \sum_{x \in V} f(x) \cdot (Lf)(x) \leq (d - \lambda) \sum_{x \in V^+} g^2(x) = \tag{1}$$

$$= (d - \lambda) \sum_{x \in V} f^2(x) = (d - \lambda) \| f \|^2$$

From claim 5 we have

$$B_f \leq \sqrt{2d} \cdot \| fM \| \cdot \| f \|$$

and from claim 6 we have

$$B_f \geq h(G) \| f \|^2$$

If we combine these results we get

$$h(G) \| f \|^2 \leq \sqrt{2d} \cdot \| fM \| \cdot \| f \|$$

If we square this equation and combine with (1) we get

$$h^2(G) \| f \|^2 \leq 2d \cdot \| fM \|^2 \leq 2d(d - \lambda) \| f \|^2$$

and therefore

$$h^2(G) \leq 2d(d - \lambda)$$

□

3 The Infinite d-Regular Tree

Let us look at the infinite adjacency matrix A_T of the infinite d-regular tree T. The infinite vectors we work with are those in $l_2(V(T))$:

$$l_2(V(T)) = \{x : V(T) \to \mathbb{R} | \sum_{v \in V(T)} x_v^2 < \infty\}$$

Define $\sigma(A_T)$ to be the set of all λ's such that $(A_T - \lambda I)$ is non invertible. This is the set of all λ's such that $(A_T - \lambda I)$ is not one to one, i.e. there is a vector $u \in l_2$ such that $(A_T - \lambda I)u = 0$, or $(A_T - \lambda I)$ is not onto l_2.

Theorem 7.

$$\sigma(A_T) = [-2\sqrt{d-1}, 2\sqrt{d-1}]$$

Given $v \in V(T)$ (the root of the tree),

$$\lambda \in \sigma(A) \iff \delta_v \notin Range(\lambda I - A)$$

where δ_v is defined as follows:

$$\delta_v(u) = \begin{cases} 1 & u = v \\ 0 & u \neq v \end{cases}$$

The direction $\Leftarrow$ is easy. The other direction requires a proof which will not be given here.

We wish to find out for which values of λ, $\delta_v \in Range(\lambda I - A_T)$.

We are trying to find a function $f \in l_2$ such that

$$\delta_v = (\lambda I - A)f \tag{2}$$

Without loss of generality, f is spherical, meaning if u, w are the same distance from v then $f(u) = f(w)$. This is true since if g is a solution to (2) then so is f which is the spherical symmetrization of g (for all vertices with a given distance from v, f will be the average of g on these vertices). Therefore, $f(u)$ depends only on the distance $d_T(u, v)$. We need to define a sequence of numbers $x_0, x_1, \ldots$ such that all vertices u with $d_T(u, v) = r$ will have $f(u) = x_r$. Substituting the sequence $\{x_i\}_{i=0}^{\infty}$ for f in (2), we get the following recursion:

$$\lambda x_0 = d x_1 + 1$$

$$\lambda x_i = x_{i-1} + (d-1)x_{i+1}$$

We will try to find two numbers ρ_1, ρ_2, such that for every i, $x_i = A\rho_1^i + B\rho_2^i$. To find these numbers, we need to solve the equation

$$\lambda \rho = 1 + (d-1)\rho^2$$

The solutions to this equation are

$$\rho_{1,2} = \frac{1}{2(d-1)}(\lambda \pm \sqrt{\lambda^2 - 4(d-1)})$$

If $\lambda^2 < 4(d-1)$ ($\lambda \in \sigma(A)$) then $\rho_{1,2}$ are complex and $|\rho_1| = |\rho_2| = \frac{1}{\sqrt{d-1}}$. In this case, f is not in l_2: $|x_i| = \Theta((d-1)^{-\frac{i}{2}})$, and for every i x_i appears $\Theta((d-1)^i)$ times in $\| f \|$, each time contributing $\Theta(((d-1)^{-\frac{i}{2}})^2)$ to the sum. This means that each level i in the tree contributes $\Theta(1)$ to the sum, which results in the sum being infinite.

On the other hand, if $\lambda^2 > 4(d-1)$, then one of the roots ρ_1, ρ_2, say ρ_1, is less than $\frac{1}{\sqrt{d-1}}$ in absolute value, in which case we can choose $B = 0$, so the contribution of the i'th level of the tree to f's norm will be exponentially small in i, and therefore $f \in l_2$. Also, there is a solution to $\lambda A = dA\rho + 1$, since $\lambda \neq d\rho$:

$$\lambda \stackrel{?}{=} \frac{d}{2(d-1)}(\lambda - \sqrt{\lambda^2 - 4(d-1)})$$

$$\sqrt{1 - \frac{4(d-1)}{\lambda^2}} \stackrel{?}{=} 1 - \frac{2(d-1)}{d}$$

This equality cannot hold for $d > 2$, since the r.h.s. is negative.