Expander Graphs and their Applications*
Lecture 4: a Geometric View of Expander Graphs

Notes taken by Eran Ofek and Erez Waisbard

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Summary: In the previous lectures we dealt with expander graphs in the combinatorial aspect (algorithmic and complexity applications), the algebraic aspect (spectral gap) and probabilistic aspect (rapidly mixing random walks). In this lecture we start dealing with the geometric/differential aspect of expander graphs. We introduce the construction of Margulis for expander graphs which is in fact a continuous graph with an expansion property. We show an analogy between expansion in graphs and the Cheeger constant which is defined for Riemannian surfaces. We also show the connection between the expansion constant and the spectral gap.

1 The Classical Isoperimetric Problem

A very natural (and ancient) question in geometry is the following:
Of all simple closed curves in the plane of a given length, which curve encloses the greatest area?

The solution to this question is obviously a circle. Although this fact was already known to the Greeks, they could not prove it. The first proof for this fact (which can be considered rigorous) is the proof of Jacob Steiner (1800's). This proof uses a method called Steiner Symmetrization. We will briefly sketch the idea of this method. Let K be a closed plane curve, let l be a line in R^2. The *symmetrization* of K with respect to l which we denote by K^* is a region in R^2 which is symmetric about l such that any line perpendicular to l intersects K iff it intersects K^*, and the intersections have the same length; furthermore the intersections of lines perpendicular to l with K^* are connected. It can shown (via calculus) that K^* has the same area as K and it's boundary length does not increase with respect to K. In fact, if l is not parallel to a line of symmetry of K then symmetrization decreases boundary length.
To gain some intuition for the correctness of this statement, we will show it for the special

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case in which K is a polygon which is composed of parallel trapezoids as demonstrated in figure 1.

In this case K^* is accepted from K if we transform each trapezoid into a symmetric trapezoid (a trapezoid of equal sides length) with the same bases and height as illustrated in figure 1. The area of the new trapezoid remains the same. Furthermore, the sum of the side lengths can only decrease (this fact can be easily verified). It follows that the boundary length of K^* is less or equal to that of K.

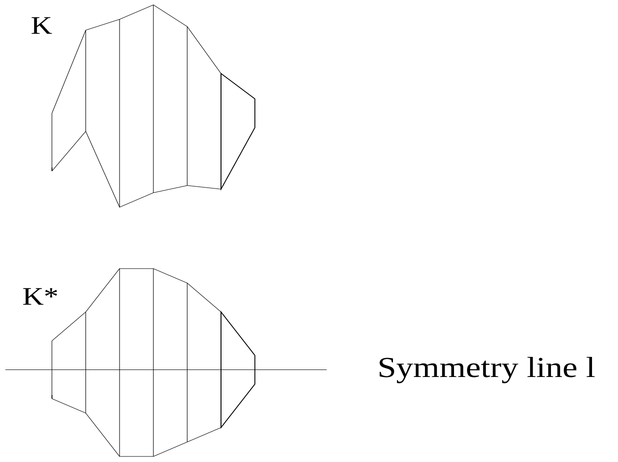

Figure 1: Steiner Symmetrization

We remark that the area and boundary length of any closed curve is accepted by considering the limit of the area and boundary length of shapes of this special form (see figure 2). This gives intuition for the correctness of the claim in general case.

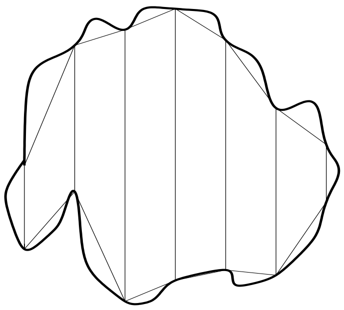

Figure 2:

2 Graph Isoperimetric problems

In the spirit of the last section, one can define an analogous problem in graphs (rather than in the Euclidean space). The graph is analogous to the plane, closed curves are analogous to subsets of vertices, the "area" of a subset of vertices is it's cardinality and the "boundary length" of a subset is the number of edges which go out from it (or the vertices of these edges which are outside the set). More specifically we define the following isoperimetric problems:

Definition 1. *The edge isoperimetric problem*: given a graph G and a number k find

$$\Phi_E(G,k) = \min_{S \subset V}\{|E(S,\overline{S})| \ : \ |S| = k\}$$

Definition 2. *The vertex isoperimetric problem*: given a graph G and a number k find

$$\Phi_V(G,k) = \min_{S \subset V}\{|\Gamma(S) \setminus S| \ : \ |S| = k\}$$

2.1 The discrete cube

Let us consider first a well known graph for which the isoperimetric problem is partially solved. The discrete cube graph G_d is formally defined as:

$$V(G_d) = \{0,1\}^d$$

$$E(G_d) = \{(v_1,v_2): \ v_1,v_2 \in \{0,1\}^d, \text{the Hamming distance between } v_1,v_2 \text{ is } 1\}$$

An equivalent definition for the d-dimensional cube graph is by recursion:

- G_1 equals K_2 (i.e. two vertices connected by an edge).
- G_{d+1} is accepted by taking two copies of G_d and connecting vertex i in the first copy to vertex i in the second copy (for all i).

It is known that if $k = 2^l$ then $\Phi_E(G_d,k)$ is achieved by a set of 2^l vertices which induces an l-dimensional cube. In this case $\Phi_E(G_d,2^l) = 2^l(n-l)$.

For $k = \binom{d}{0} + \binom{d}{1} + ... + \binom{d}{r}$ the vertex expansion $\Phi_V(G_d,k)$ is achieved by any set S which is a ball of radius r around some vertex v_0:

$$S = \{v: \ v \in \{0,1\}^d, \text{The Hamming distance between } v,v_0 \le r\}$$

3 The construction of Margulis, Gabber-Galil

In this section we describe one of the first explicit construction of an expander graph. In contrast to the expanders we encountered so far in this course, the construction they give is over a continuous set.

We denote by I the interval $(0,1)$. The set of vertices is all the points in the continuous cube $I \times I$. Two linear transformations define the edges:

$$T(x,y) \to (x+y,y) \bmod 1$$

$S(x,y) \to (x, x+y) \bmod 1$

The neighbors of a point (x,y) are the points: $T(x,y), S(x,y), T^{-1}(x,y), S^{-1}(x,y)$. Thus the graph is 4-regular. The expansion property of this graph is described by the following theorem:

Theorem 3. ***(Margulis,Gabber-Galil)***
There exists some $\epsilon > 0$ such that for any measurable set $A \subset I \times I$ with $\mu(A) \leq \frac{1}{2}$ (μ denotes the Lebegue measure) the following holds:

$$\mu(\Gamma(A) \cup A) \geq (1+\epsilon)\mu(A),$$

where $\Gamma(A) = S(A) \cup T(A) \cup S^{-1}(A) \cup T^{-1}(A)$ is the set of all points which are neighbors of points in A.

It s worthwile to mention here the following conjecture:

Conjecture 4. *(Linial) For any measurable subset A, such that $\mu(A) \leq \frac{1}{2}$,*

$$\mu(A \cup S(A) \cup T(A)) \geq \frac{4}{3}\mu(A),$$

with equality achieved by the hexagon whose vertices are $(0,a)$, $(a,0)$, $(a,-a)$, $(0,-a)$, $(-a,0)$, $(-a,a)$ for some small $a > 0$.

4 The Cheeger constant, Cheeger inequality

In this section we introduce the Cheeger constant. Loosely speaking, this constant represents the "expansion" of a curve.

Definition 5. Let $M^{(n)}$ be an n-dimensional Riemann surface. The **Cheeger constant** of M is defined to be:

$$h(M) = \min_{\substack{C \text{ is an} \\ (n-1)\text{-dimensional} \\ \text{surface which divides} \\ M \text{ into } M_1, \ldots M_t}} \frac{\mu_{n-1}(C)}{\min_i \ \mu_n(M_i)}$$

where $\mu_{n-1}(C)$ is the area of C and $\mu_n(M_i)$ is the volume of M_i.
An intuitive demonstration of the definition is illustrated in figure 3
The analogy between the Cheeger constant and expansion is as follows:
$M \sim G$,
C is analogous to a cut in G,
M_1, M_2 are analogous to $S, \overline{S}$,
$\mu_{n-1}(C) \sim |e(S, \overline{S})|$,
$\min_i \mu_n(M_i) \sim \min\{|S|, |\overline{S}|\}$.

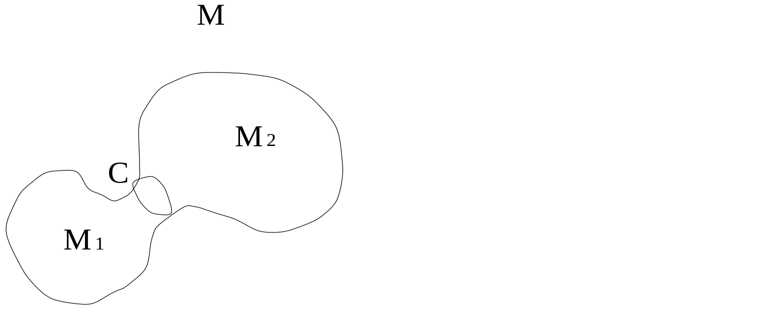

Figure 3: C divides M into M_1, M_2

Given an n-dimensional Riemann surface $M(n)$, and a function $f : M(n) \to \mathbb{R}$, then its Laplacian is $\Delta(f) = \mathrm{div}(\mathrm{grad}(f))$. The Laplacian is a linear operator, and its eigenvalues are all the numbers λ, for which there is a function $f : M(n) \to \mathbb{R}$ satisfying $\Delta f = \lambda f$. All its eigenvalues are non-negative, and its lowest eigenvalue is zero, corresponding to the constant function.

Theorem 6. *Let M be a Riemann surface as described before, denote by λ the lowest positive eigenvalue of the Laplacian of M, then $\lambda \geq \frac{h^2}{4}$.*

We will now explain the discrete analogs for the gradient and divergence operators in graphs. Let $G = (V, E)$ be an undirected graph. Select an orientation for the edges of G. Let M be the $V \times E$ adjacency matrix of G where the entry $M_{v,e}$ equals 1 (-1) if the edge e enters (leaves) v and 0 otherwise.

The gradient: Let $f : V \to R$ be a function on the vertices of G. f can be thought of as a row vector with V entries. The gradient operator is $f \mapsto fM$. The gradient of f is a vector with E entries which tells us how does f change along the edges of the graphs. I.e., if e is the edge from u to v, then $(fM)_e = f_v - f_u$.

The divergence: Let $g : E \to R$ be a function on the edges of G. g is a column vector with E entries. The divergence operator is $g \mapsto Mg$. The divergence of g is a V dimensional vector with $Mg_v = \sum_{\text{e enters v}} g(e) - \sum_{\text{e leaves v}} g(e)$.

The Laplacian: If we go through with the analogy between real functions in R^n and functions on the vertices of a graph, then the discrete analog of the Laplacian will be: $f \mapsto MM^t f$ (for $f : V \to R$). The matrix $L = MM^t$ is called the Laplacian of G. A simple calculation shows that L is the following symmetric matrix:

$$L_{i,j} = \begin{cases} -1, & (i,j) \in E \\ \deg(i), & i = j \end{cases}$$

In the case that G is a d-regular graph (with A_G it's corresponding adjacency matrix):

- $L = d \cdot I - A_G$.

- The spectrum of L is in $[0, +2d]$ (since the spectrum of A_G is in $[-d, +d]$).
- $\lambda_1(A_G) = d$ corresponds to $\lambda_n(L) = 0$ and in general $\lambda_i(A_G) = d - \lambda_{n-i+1}(L)$.
- The spectral gap of G ($\lambda_1(A_G) - \lambda_2(A_G)$) equals to the lowest positive eigenvalue of L.

Notice the similarity between Theorem 6 and the upper bound on h given by Theorem 9.

5 Expansion and the spectral gap

In this section we show the that a graph has high expansion (high Cheeger constant) iff it has a large spectral gap.

5.1 The Rayleigh quotient

For a real symmetric matrix A one can obtain the eigenvalues of A using a special quotient known as the Rayleigh quotient.

Theorem 7. *Let A be a real symmetric matrix and let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be its corresponding eigenvalues. Then:*

$$\lambda_1 = \max_{||x||=1} \frac{xAx^t}{||x||}, \qquad \lambda_2 = \max_{||x||=1, x \perp x_1} \frac{xAx^t}{||x||} \qquad \ldots \qquad \lambda_n = \max_{||x||=1, x\perp x_1, \ldots, x \perp x_{n-1}} \frac{xAx^t}{||x||}$$

where x_i is an eigenvector corresponding to λ_i.

5.2 The main theorem

Before stating the theorem, we formally define the expansion constant:

Definition 8. The expansion constant of a graph $G = (V, E)$, is

$$h(G) = min_{S \subseteq V, |S| \leq \frac{|V|}{2}} \frac{|E(S, \overline{S})|}{|S|}$$

Theorem 9. *Let $G = (V, E)$ be a finite, connected, k-regular graph without loops. Let λ be the second eigenvalue of G. Then*

$$\frac{k - \lambda}{2} \leq h(G) \leq \sqrt{2k(k - \lambda)}$$

Proof. In this lecture we only prove the lower bound on h, showing that a large gap implies high expansion. In order to prove that $\lambda \geq \ k - 2h(G)$, we will give a vector $f \perp \vec{1}$ for which $\frac{fAf^t}{||f||} \geq k - 2h(G)$. For $S \subset V$ we define f to be the following weighted cut function:

$$f = |\overline{S}| 1_S - |S| 1_{\overline{S}}$$

using the Rayleigh quotient we get:

$$\lambda \geq \frac{fAf^t}{||f||^2}.$$

We will evaluate the rhs. Starting with the denominator we get that:

$$||f||^2 = |\overline{S}|^2|S| + |S|^2|\overline{S}| = |S||\overline{S}|(|S| + |\overline{S}|) = n|S||\overline{S}|$$

moving on to the numerator we get:

$$fAf^t = 2|E(S)||\overline{S}|^2 + 2|E(\overline{S})||S|^2 - 2|S||\overline{S}||E(S,\overline{S})| \tag{1}$$

Since G is a k-regular graph

$$k|S| = 2|E(S)| + |E(S,\overline{S})|$$

and

$$k|\overline{S}| = 2|E(\overline{S})| + |E(S,\overline{S})|$$

substituting $2|E(S)|$ and $2|E(\overline{S})|$ in (1) yields:

$$fAf^t = nk|S||\overline{S}| - n^2|E(S,\overline{S})|$$

We now plug it in and get

$$\lambda \geq \frac{fAf^t}{||f||^2} = \frac{nk|S||\overline{S}| - n^2|E(S,\overline{S})|}{n|S||\overline{S}|} = k - \frac{n|E(S,\overline{S})|}{|S||\overline{S}|}$$

Fix S to be a set for which

$$h(G) = \frac{|E(S,\overline{S})|}{|S|}$$

it follows that:

$$\lambda \geq k - \frac{nh(G)}{|\overline{S}|} \geq k - 2h(G)$$

(since $|\overline{S}| \geq \frac{n}{2}$). $\square$