Expander Course - Homework set 9

Let $G = (V_L; V_R, E)$ be a bipirtite k-regular graph with n vertices on each side ($|V_L| = |V_R| = n$), and let λ be an upper bound on its normalized second eigenvalue in absolute value. Also, let $C_0 \subseteq \{0,1\}^k$ be a linear code with $\mathrm{dist}(C_0) = \delta_0 k$ and $\mathrm{rate}(C_0) = R_0$.

Let $N = kn = |E|$, and define the code $C \subseteq \{0,1\}^N$ as follows. We regard a vector $x \in \{0,1\}^N$ as a zero-one assignment to the edges, and for each vertex v we fix some ordering of the k edges incident with it. Furthermore, we define $x_v \in \{0,1\}^k$ to be the "ordered" restriction of x to the edges incident with v. The code C is defined by:

$$C = \{x \in \{0,1\}^N : x_v \in C_0 \text{ for all } v \in V_L \cup V_R\}.$$

We denote $R = \mathrm{rate}(C)$ and $\delta = \mathrm{dist}(C)/N$.

Problem 1. Prove that $R \geq 2R_0 - 1$.

Problem 2. Prove that $\delta \geq \delta_0 \cdot (\delta_0 - \lambda)$. Hint: Define A_L (resp. A_R) as the set of left (resp. right) vertices that "see" at least $\delta_0 k$ errors in a minimal weight codeword.

Problem 3. Consider the following decoding algorithm. Given an input vector $x \in \{0,1\}^N$, on the first iteration, in parallel, each left vertex $v \in V_L$ decodes its input x_v as the closest codeword in C_0 and updates the vector x. On the second iteration, each vertex on the right decodes and updates. The algorithm proceeds by left and right decoding alternately, and outputs x after a predetermined number of iterations.

Prove that if there is a codeword $c \in C$ at distance ηN of x, where

$$\eta < \frac{\delta_0}{2} \cdot (\frac{\delta_0}{2} - \lambda),$$

the algorithm will output c in $O(\log n)$ iterations. (We think here of δ_0, λ and η as fixed, and growing n).

Hint: Define $A_0 \subseteq V_L$ as the set of left vertices that "see" at least $k\delta_0/2$ errors in the input vector x, $A_1 \subseteq V_R$ as the set of right vertices that see too many errors after the first iteration, and so on. Argue that if there are not too many errors, the size of these sets must decrease exponentially.