Expander Course - Homework set 8

Due: May 27

Problem 1. What happens if in the variational definition for eigenvalues, we replace the l_2 norm by l_∞? Show that if G is a d-regular graph and $B = A/dn - J/n^2$, where A is G's adjacency matrix and J is the all ones matrix, then

$$\max \frac{xBy}{\|x\|_\infty \cdot \|y\|_\infty}$$

equals G's *edge discrepancy number*. Namely,

$$\max \left|\frac{e(S,T)}{dn} - \frac{|S||T|}{n^2}\right|.$$

Here the maximum is over all nonempty sets of vertices S and T (not necessarily disjoint). Also, as usual, $e(S,T)$ is the number of adjacent pairs (x, y) with $x \in S$ and $y \in T$. What can you say about

$$\frac{1}{dn} \max \left|\frac{xAy}{\|x\|_\infty \cdot \|y\|_\infty}\right|,$$

where the max is taken over all x, y orthogonal to the all ones vector?

Problem 2. The Petersen graph has ten vertices which are in 1:1 correspondence with all two-element subsets of $\{1, 2, 3, 4, 5\}$. Two vertices are adjacent if the corresponding subsets are disjoint. Determine the Euclidean distortion of this graph. **Hint:** Use symmetry considerations to reduce the space of relevant PSD matrices. Prove that this matrix can be taken symmetric w.l.o.g. Then find the optimal Q.

Problem 3. Show that the following matrix is positive semidefinite. $(d - 1)I - A + P$. All matrices are $2^d \times 2^d$. As usual, I is the identity matrix; A is the adjacency matrix of the d-dimensional cube, and P is a permutation matrix whose (x, y) entry is one iff x and y are two antipodal vertices of the d-cube. **hint:** Prove that the eigenvectors of A (which were calculated earlier in this course) are also eigenvectors of this matrix.