Expander Course - Homework set 7

Due: May 13

We identify undirected graphs (possibly with parallel edges and self loops) as symmetric integer matrices, so e.g. for a graph G, define G^k by raising G's adjacency matrix to the k'th power. G is d-regular if the row-sum is d for every row. G is an (n, d, α)-graph if it has n vertices, it is d-regular, and $\lambda(G) \leq \alpha d$. Note that $\alpha \in [0, 1]$. We sometimes call it an α-expander.

A (possibly partial) mapping from graphs to graphs is expansion preserving if it always maps α-expanders to $f(\alpha)$-expanders, for some real function f taking $[0, 1)$ to $[0, 1)$. The mapping is strongly expansion preserving if $f(\alpha)$ goes to zero with α. Similarly we define expansion preserving mapping and its strong version for (partial) mappings from pairs of graphs to graphs. Note that we proved in class that the zigzag product is a strongly expansion preserving mapping of pairs of graphs to graphs.

Problem 1.

1. We have already seen $G \to G^k$ preserves expansion. Prove that the inverse map (whenever defined) also preserves expansion for every fixed k.

2. Prove that for every fixed $c > 1$, every map sending a d-regular graph to a t-regular subgraph or supergraph of G, with $d/c < t < cd$ is expansion preserving.

3. Analyze the following variant of zigzag product, called *replacement product*. Let G be an (n, m, α)-graph, and H be an (m, d, β)-graph. Let $G \textcircled{r} H$ denote the following graph, constructed in two steps. First, replace every vertex of G by copy of H on the endpoints of its incident edges (in an arbitrary way). So there are mn vertices, like in zigzag. Unlike zigzag, we keep all edges of both G and H (so far the graph has degree $d+1$; in every vertex we have d edges from H and one from G). Next, (to balance the contribution of edges from G and H), we split every original edge of G into d parallel edges (so now the degree is $2d$). This completes the definition. Prove that $\textcircled{r}$ is an expansion preserving mapping (by providing an expansion preserving mapping from $G \textcircled{z} H$ to $G \textcircled{r} H$), but it is not strongly expansion preserving.

Recall that a sequence of graphs G_n is *explicit* if there is a polynomial time algorithm that on input 1^n outputs the adjacency matrix of G_n.
Recall that G_n is *strongly explicit* if there is a polynomial time algorithm that on input $(1^n, v, k)$ computes the kth neighbour of v in G_n (whenever the binary string v encodes a vertex of G_n which has at least k neighbours).

Problem 2.

1. Describe all eigenvalues of $G \otimes H$, given the eigenvalues of G and H. Prove that $\otimes$ is strongly expansion preserving.

2. In class we defined the following sequence of graphs G_n is an expander family. Start form any fixed $(d^4, d, 1/5)$-graph H.
Define $G_1 = H^2$ and
$$G_{t+1} = G_t^2 \textcircled{z} H$$
Explain why G_n is an explicit family.

3. The family G_n above is not known to be strongly explicit. Here is a modification which fixes it. Start with any fixed $(d^8, d, 1/10)$-graph H. Define $G_1 = H^2$, $G_2 = H \otimes H$, and
$$G_t = \left(G_{\lceil \frac{t-1}{2} \rceil} \otimes G_{\lfloor \frac{t-1}{2} \rfloor}\right)^2 \textcircled{z} H.$$
Prove that G_n is a strongly explicit family. Prove that it is an expander family.

Problem 3. Here we deal with constructing the base graphs for the iterative constructions above.

1. Assume K is an (q^2, q, α)-graph for a sufficiently small constant α. Show how to construct the two base graphs H needed in the above problem using a constant number of power, tensor and zigzag operations.

2. Show that for every prime power q the following (Affine Plane) graph K is a $(q^2, q, 1/\sqrt{q})$-graph. The vertices of K are $GF(q^2)$. Connect (a, b) and (c, d) by an edge iff $ac = b + d$.
Hint: Study the eigenvalues of K^2, which is an extremely simple matrix.

Problem 4. Try to find a lower bound on the 2nd eigenvalue of the Margulis graph, which is better than the Alon-Boppana bound, by giving an appropriate "test function".