Expander Course - Homework set 6

Due: May 6

Today's homework set covers some of the salient features of random regular graphs. This is a large, active and fascinating subject. A good coverage can be found in the book "Random Graphs" by Janson, Luczak, and Rucinski. A more updated and more comprehensive coverage can be found in a survey article by Nick Wormald "Models of random regular graphs" which can be downloaded from his home page.

Problem 1. How to generate random d-regular graphs: Consider the following distribution on graphs. You start with n sets of d vertices each (*clusters*). You select a random matching on these dn vertices, and contract each cluster to a vertex. What you obtain need not be a simple graph, since loops and parallel edges may have been created. If it is a simple graph, you accept it, otherwise you reject it. Prove:

- This process generates d-regular graphs with uniform distribution.
- For $d = 2$ (this is an uninteresting case for us, but the general question is a little too hard for an exercise for the uninitiated), show that that the probability of acceptance (i.e. a simple graph is generated) is bounded away from zero. What is this probability? Read about the general case in the above ref's, or in Alon and Spencer, the chapter about the Poisson paradigm.

Problem 2. For k even, we define the *double factorial* $k!! = (k-1)(k-3)...3 \cdot 1$. Observe that this is the number of perfect matchings on k distinct vertices. Prove that $k!! = \Theta((k/e)^k)$.

Problem 3. Show that there is an $\epsilon > 0$, such that almost all 3-regular graphs are ϵ-expanders. Namely, a randomly generated 3-regular graph of order n is an ϵ-expander with probability that tends to 1 as n grows.