Expander Course - Homework set 5

Due: April 29

Problem 1. Let G be a d-regular graph of girth g, and let A be G's adjacency matrix. Let $A^{(t)}$ be the matrix of distance t at G. Namely, the (x, y) entry in $A^{(t)}$ is 1 if the vertices x and y are at distance t in G and zero otherwise. For completeness sake, we mention that $A^{(0)} = I$, the identity matrix, and $A^{(1)} = A$. Prove that there exist real polynomials $P_t(\cdot)$ so that $A^{(t)} = P_t(A)$. Write the recurrence relation for these polynomials, show that (i) P_t has degree t, (ii) All t roots of P_t are real (iii) All roots are in the interval $[-2\sqrt{d-1}, 2\sqrt{d-1}]$. **Hint:** These polynomials are called Geronimus Polynomials and they (possibly slightly modified) appear in the literature. Many of their properties can be viewed in the more general context of orthogonal polynomials on which there is a vast literature.

Problem 2. Given integers d and h, determine the spectrum of the d-regular tree of height h. **Warning:** This problem appears in the Lovasz problem book. Unfortunately, this is one of the rare cases where there is a (slight, but annoying) error in the book.

Problem 3. As we observed in class, while expansion implies eigenvalue separation, the bound on the eigenvalue gap is rather weak. In this exercises you are requested to show that better estimates would indeed require stronger assumption. Given $\epsilon > 0$, show that for large enough d, there exist arbitrarily large d-regular graphs with edge-expansion $> (1 - 2\epsilon)d$ whose second eigenvalue is $> \epsilon d$. **Hint:** Construct a random $(1 - \epsilon)d$-regular graph and add to it a disjoint collection of cliques of size ϵd each. The random graph will provide the expansion, and the second eigenvalue cannot be too small by eigenvalue interlacing.