Expander Course - Homework set 4

Due: April 22

Problem 1. In this problem you are requested to prove some version of the edge-isoperimetric theorem for the cube. Show that if S is a subset of the n-dimensional cube $\{0,1\}^n$, then $e(S)$, the number of edges with both vertices in S does not exceed

$$\frac{1}{2}|S|\log_2 |S|.$$

The bound is obviously tight, but can you characterize the case of equality? Note also that n does not appear in the bound. **Hint:** Induction on n. You may find it useful to show that $H(x) \geq 2x$ for $\frac{1}{2} \geq x \geq 0$, where H is the binary entropy function.

Problem 2. This problem concerns a weaker version of the vertex-isoperimetric theorem for the cube, along the following line of arguments:

1. Let $r \geq k > 0$ be integers, u, v, w real with $u \geq v \geq r-1$, and $u \geq w \geq r-2$. Suppose that

   $$\binom{u}{r} = \binom{v}{r} + \binom{w}{r-1}.$$

   Show that

   $$\binom{u}{k} \leq \binom{v}{k} + \binom{w}{k-1}.$$

2. Let S be a family of r-subsets of $[n]$. A set $B \subseteq [n]$ is in the k-*shadow* of S if $|B| = k$, and there is some $A \in S$ that contains B. Let $|S| = \binom{x}{r}$ (here $x > 0$ is real, not necessarily an integer). Show that there are at least $\binom{x}{k}$ sets in the k-shadow of S.

3. Finally show that if $F \subseteq \{0,1\}^n$, has cardinality $\sum_0^r \binom{n}{j}$, then there are at least $\binom{n}{r+1}$ vertices in F's vertex boundary (vertices not in F that have a neighbor in F).

Problem 3. Prove the Matrix-Tree theorem. Let G be a graph and let L be the Laplacian matrix of G. Let M be the matrix that is obtained by deleting the i-th row and the i-column of L, for some i. Show that $\det(M)$ equals the number of spanning trees in G. **Hint:** We noted in class that $L = AA^t$ where A is the incidence matrix of (an orientation of) G. Determinants of such matrices can be computed by means of the Binet-Cauchy identity: Let A be an $n \times m$ matrix. Then

$$\det(AA^T) = \sum_I (\det(A_I))^2.$$

The summation is over all subsets $I \subseteq [m]$ of cardinality n, and A_I is the submatrix of A where you leave in only the columns whose index is in I.