Expander Course - Homework set 1

Due: April 8 (first class after Pesach)
(for this and another homework set to be given next week)

All graphs in this problem set are undirected, connected and have no self loops (but may have parallel edges).

Definition 1. *For a graph G, we let*

- *$Diam(G)$, the diameter of G, be the distance of the two furthest vertices in G.*
- *$i(G)$, the independence number, be the size of the largest set of vertices containing no edges.*
- *$\chi(G)$, the chromatic number, be the minimum number of independent sets covering all vertices.*
- *$Girth(G)$, the girth, is the length of the shortest cycle in G.*

We denote d-regular graphs on n vertices by (n, d)-graphs.

Problem 1. *Prove that for every n, d and every (n, d)-graph*

1. *$Diam(G) \geq \log_{d-1} n$*
2. *$\chi(G) \leq d + 1$*

Problem 2. *Show that a for every $n, d \geq 2$ and random $(n, 2d)$-graph*[1] *almost achieves these bounds: For some constants a, b independent of n, d with positive probability it holds that*

1. *$Diam(G) \leq a \log_d n$*
2. *$\chi(G) \geq bd/ \log d$*

[1] It is hard to generate such graphs and argue about them - instead you can use the random graph model in which every vertex picks d neighbours at random (note that this graph will have the same number of edges, but will typically not be regular

Problem 3. Bonus

1. *Prove (via a probabilistic argument) the existence of graphs of arbitrarily high girth and chromatic number simultaneously (namely for every integer k prove the existence of a graph whose girth and chromatic number are at least k).*

2. *Explicitly construct graphs with no triangles, and arbitrarily high chromatic number.*

Problem 4. *The standard random walk on a graph G moves a token on its vertices according to the following rule. If the token is currently on vertex v, then at the next step it remains at v with probability $1/2$, and with the remaining $1/2$ probability it moves to a uniformly chosen neighbour of v in G. Let P_0 be some initial distribution, P_t be the distribution after t steps. Let U be the uniform distribution on the vertices.*

The statistical (or variational) distance $v(P, Q)$ between two probability distributions P, Q on the same set S is 1/2 their L_1 distance, namely $\frac{1}{2}\sum_{s\in S}|P(s) - Q(s)|$.

$Mix(G)$, the mixing time on G, is defined to be the smallest t for which $v(P_t, U) = \frac{1}{2}\|P_t - U\|_1 \leq 1/4$, when we maximize over all initial distributions P_0. (Recall that $\|\vec{v}\|_1 = \sum_{i=1}^{n}|v_i|$.)

1. *Prove that on every (n, d)-graph the standard random walk converges to the uniform distribution from every starting distribution P_0.*

2. *Prove that the mixing time is maximized when P_0 is supported on one vertex.*

3. *Prove that if $Mix(G) = T$, then $v(P_{kT}, U) \leq 2^{-k}$*

Problem 5. *Consider a random bipartite graph on two vertex sets L, R of size n each, in which every vertex in L independently picks d uniformly independent random neighbours in R. Given $\epsilon > 1/d$, prove that with high probability, every set S in L of size $\alpha n/d$ has at least $|S|(1 - \epsilon)d$ neighbours in R, where α depends only on ϵ.*